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A Note on A New Improved Poisson Distribution and Its Approximations

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Abstract

This work, determine a new improved Poisson distribution with mean $\mu = \frac{nr}{b}$ *from Polya distribution with parameters* r, b, n *and* c *where* r, b, n *and* c *are non-negative integer. It was found that the new improved Poisson approximates binomial distribution accurately more than the normal Poisson for* n *so large and* $\frac{r}{r+b}$ *sufficient small for* $x \in \{0,1 \dots n\}$ *while the new improved Poisson approximates Polya distribution for a certain random variables* $x \in \{0.1\}$, *provided that* n *is not large enough. This implies the new improved Poisson distribution is not sufficient to enough to approximate Polya distribution.*

Keywords*- Binomial distribution, Poisson distribution, A new Improved Poisson*

1.0 INTRODUCTION

Let X be a non- negative integer –valued random variable such that $0 \leq x \leq n$ is of the form

$$P_Y(x) = \frac{\binom{\frac{r}{c}+x-1}{x}\binom{\frac{b}{c}+n-x-1}{n-x}}{\binom{\frac{r+b}{c}+n-1}{n}} \quad x = 0,1, \dots \dots n \qquad 1.0$$

The mean and variance is given as $\frac{rn}{r+b}$ and $\frac{nrb(r+b+cn)}{(r+b)^2(r+b+c)}$ respectively

The probability function in (1.0) can be expressed as of the form

$$P_Y(x) = \binom{n}{x}\frac{(r.c)_{x-1}(b,c)_{n-x-1}}{(r+b,c)_{n-1}} \quad x = 0,1 \dots . n \qquad 1.1$$

Let X be a non –negative valued random variable that has a Binomial distribution with parameters n, and $\frac{r}{r+b}$. Its probability function can be expressed as

$$B\,(x)_{n,\frac{r}{r+b}} = \binom{n}{x}\frac{r^x b^{n-x}}{(r+b)^n} \quad x = 0,1 \dots n \qquad 1.2$$

By a taking the limit as $r, r+b \to \infty$ as $\frac{r}{r+b}$ remain constant $P_Y(x) \to B\,(x)_{n,\frac{r}{r+b}} = \binom{n}{x}\frac{r^x b^{n-x}}{(r+b)^n}$ $x = 0,1 \ldots n$

And also if $n \to \infty, \frac{r}{r+b} \to 0$ and $\mu = n\frac{r}{r+b}$ remain constant then $B\,(x)_{n,\frac{r}{r+b}} = \binom{n}{x}\frac{r^x b^{n-x}}{(r+b)^n} \to \wp_\lambda(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $x = 0,1 \ldots n$

Samson et al [1], [4] gave a new Improved Poisson in extension of Teerapabolarn [8] with mean $\frac{NA}{B}$ by deriving from the Generalized Binomial distribution as of the form

$$G_{bd}(A,B,N) \cong \frac{\wp_\lambda(x)\left(\frac{B}{A+B}\right)^N e^\lambda}{1+\frac{x(x-1)}{2N}} \qquad 1.3$$

Where $\wp_\lambda(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ Samson et al [1], [4] used (1.3) to generate a model for determing the prices of options.

This work , focus on determing a new improved Poisson from Polya distribution with $\mu = \frac{rn}{b}$ for approximation of Polya distribution and Binomial distribution respectively , in extension of Samson et al [1] ,[4].The level of accuracy is given in the point metric form .

2.0 METHOD

The tools for giving the result are Polya distribution. Polya distribution used in this study was discussed by Teerapabolarn [11].It is a discrete distribution that depends on four parameters N, n, r and c where N, n, r and $c \in \mathbb{N}$ and the mean and the variance of X are $\mu = \frac{nr}{r+b}$ and $\sigma^2 = \frac{nrb(r+b+cn)}{(r+b)^2(r+b+C)}$ .The probability distribution of a random variable X taking non- negative value x, such that $0 \le x \le n$ is of the form

$$P_Y(x) = \frac{\binom{\frac{r}{c}+x-1}{x}\binom{\frac{b}{c}+n-x-1}{n-x}}{\binom{\frac{r+b}{c}+n-1}{n}} \quad x = 0,1,\ldots\ldots n \qquad 2.0$$

From (2.0) we obtained

$$P_Y(x) = \binom{n}{x}\frac{[r(r+c)\ldots.r+(x-1)c][b(b+c)\ldots b+(n-x-1)c]}{[(r+b)\ldots.r+b+(n-1)c]}$$

$$= \binom{n}{x}\frac{[r\ldots r+(x-1)c][b\ldots.b+(n-x-1)c]}{[r+b\ldots r+b+(n-1)c]}$$

$$= \binom{n}{x}\frac{\left[\frac{r}{c}\ldots\frac{r}{c}+\beta\right]\left[\frac{b}{c}\ldots\frac{b}{c}+\vartheta\right]}{\frac{r+b}{c}\ldots\frac{r+b}{c}+(n-1)}$$

where $\beta = \begin{cases} 0 \;\; if\; x = 0 \\ (x-1)\;\; if\; x = 1,2..n \end{cases}$ and $\vartheta = \begin{cases} 0\; if\; x = 0,1,..n \\ (n-x-1)\; if\; x = 0 \end{cases}$ for $c \ge 1$

$$P_Y(x) = \binom{n}{x} \frac{r+b\left[\frac{r}{r+b}\left(\frac{r}{r+b}+\frac{\beta}{\frac{r+b}{c}}\right)\right] r+b\left[\frac{b}{r+b}\cdots\frac{b}{r+b}+\frac{\vartheta}{\frac{r+b}{c}}\right]}{r+b\left[1\ldots1+\left(\frac{n-1}{\frac{r+b}{c}}\right)\right]}$$

$$= \binom{n}{x} \frac{(r+b)^x\left[\frac{r}{r+b}\left(\frac{r}{r+b}+\frac{\beta}{\frac{r+b}{c}}\right)\right](r+b)^{n-x}\left[\frac{b}{r+b}\cdots\frac{b}{r+b}+\frac{\vartheta}{\frac{r+b}{c}}\right]}{(r+b)^n\left[1\ldots1+\left(\frac{n-1}{\frac{r+b}{c}}\right)\right]}$$

$$= \binom{n}{x} \frac{(r+b)^x(r+b)^{n-x}\left[\frac{r}{r+b}\left(\frac{r}{r+b}+\frac{\beta}{\frac{r+b}{c}}\right)\right]\left[\frac{b}{r+b}\cdots\frac{b}{r+b}+\frac{\vartheta}{\frac{r+b}{c}}\right]}{(r+b)^n\left[1\ldots1+\left(\frac{n-1}{\frac{r+b}{c}}\right)\right]}$$

$$= \binom{n}{x} \frac{\left[\frac{r}{r+b}\left(\frac{r}{r+b}+\frac{\beta}{\frac{r+b}{c}}\right)\right]\left[\frac{b}{r+b}\cdots\frac{b}{r+b}+\frac{\vartheta}{\frac{r+b}{c}}\right]}{\left[1\ldots1+\left(\frac{n-1}{\frac{r+b}{c}}\right)\right]}$$

If $r, r+b \to \infty$, while $\frac{r}{r+b}$ remain constant

$$P_Y(x) \to B(x)_{n,\frac{r}{r+b}} = \binom{n}{x}\frac{r^x b^{n-x}}{(r+b)^n}$$

If X is Binomially distributed then

$$B(x)_{n,\frac{r}{r+b}} = \binom{n}{x}\frac{r^x b^{n-x}}{(r+b)^n}$$

$$= \binom{n}{x}\frac{r^x}{(r+b)^x}\cdot\frac{b^{n-x}}{(r+b)^{n-x}}$$

$$B(x)_{n,\frac{r}{r+b}} = \binom{n}{x}\left(\frac{r}{r+b}\right)^x\left(\frac{b}{r+b}\right)^{n-x} \qquad 2.1$$

Where $\lambda = \frac{nr}{r+b}$ and $\frac{r}{r+b} = \frac{\lambda}{n}$ then (2.1) becomes

$$B(x)_{n,\frac{r}{r+b}} = \binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)}{x!}\lambda^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

As $n \to \infty$

$$B(x)_{n,\frac{r}{r+b}} \to \frac{\lambda^x e^{-\lambda}}{x!} = \wp_\lambda(x)$$

This is an indication that Polya and Binomial distribution can be approximated by Poisson distribution under certain conditions on their parameters .

Lemma 2.0 Let $x \in$ N , we have the following Teerapabolarn [10]

$$\prod_{i=0}^{x-1}\left(1-\frac{i}{n}\right)=\frac{1}{1+\frac{x(x-1)}{2n}+O\left(\frac{1}{n^2}\right)}$$

3.0 RESULT

Theorem : For $x \in \mathrm{N} \cup \{0\}$ and $\mu=\frac{nr}{b}$ for $r, r+b \rightarrow \infty$ then

$$\mathrm{P}_{\Upsilon}(x) \cong \widetilde{\wp_{\lambda}(x)} \text{ and } B(x)_{n,\frac{r}{r+b}} \cong \widetilde{\wp_{\lambda}(x)} \text{ where } \widetilde{\wp_{\lambda}(x)}=\frac{\frac{\lambda^x}{x!}\left(\frac{b}{r+b}\right)^n}{1+\frac{x(x-1)}{2n}+O\left(\frac{1}{n^2}\right)}$$

Proof

For $x=0$ and $r, r+b \rightarrow \infty$

$$\mathrm{P}_{\Upsilon}(x) \cong B(x)_{n,\frac{r}{r+b}}=\binom{n}{x}\frac{r^x b^{n-x}}{(r+b)^n}$$

$$\mathrm{P}_{\Upsilon}(0) \cong B(0)_{n,\frac{r}{r+b}}=\binom{n}{0}\frac{r^0 b^{n-x}}{(r+b)^n}=\left(\frac{b}{r+b}\right)^n$$

$$\mathrm{P}_{\Upsilon}(x) \cong \binom{n}{x}\frac{r^x b^{n-x}}{(r+b)^n}$$

$$=\binom{n}{x}\left(\frac{n\frac{r}{r+b}}{n}\right)^x\left(\frac{b}{r+b}\right)^n\left(\frac{b}{r+b}\right)^{-x}=\binom{n}{x}\left(\frac{n\frac{r}{r+b}}{n}\right)^x \times \frac{\left(\frac{b}{r+b}\right)^n}{\left(\frac{b}{r+b}\right)^x}=\binom{n}{x}\frac{n^x\left(\frac{r}{r+b}\right)^x}{n^x}\frac{\left(\frac{b}{r+b}\right)^n}{\frac{\left(\frac{n\frac{r}{r+b}}{n\frac{r}{r+b}}\right)^x}{\frac{b}{r+b}}}$$

$$=\binom{n}{x}\frac{n^x\left(\frac{r}{r+b}\right)^x}{n^x}\frac{\left(\frac{b}{r+b}\right)^n}{\left(\frac{n\frac{r}{r+b}}{\lambda}\right)^x}=\binom{n}{x}\frac{n^x\left(\frac{r}{r+b}\right)^x}{n^x}\left(\frac{b}{r+b}\right)^n \times \frac{\lambda^x}{n^x\left(\frac{r}{r+b}\right)^x}$$

$$=\frac{\lambda^x}{x!}\prod_{i=0}^{x-1}\left(1-\frac{i}{N}\right)\left(\frac{b}{r+b}\right)^n=\frac{\lambda^x}{x!}\frac{1}{1+\frac{x(x-1)}{2n}+O\left(\frac{1}{n}\right)}\left(\frac{b}{r+b}\right)^n \text{ by lemma 2.0}$$

$$\mathrm{P}_{\Upsilon}(x) \cong \frac{\frac{\lambda^x}{x!}\left(\frac{b}{r+b}\right)^n}{1+\frac{x(x-1)}{2n}+O\left(\frac{1}{n^2}\right)} \cong B(x)_{n,\frac{r}{r+b}}$$

$\mathrm{P}_{\Upsilon}(x) = \widetilde{\wp_{\lambda}(x)}$, but if n is large $O\left(\frac{1}{n}\right) \approx 0$

4. NUMERICAL EXAMPLES

The following numerical examples are given to illustrate how well the new improved Poisson distribution with $\mu = \frac{nr}{b}$ approximates both Binomial and Polya distribution respectively with parameters n and $\frac{r}{r+b}$.

Example 4.1 : Suppose $n = 80,\quad \frac{r}{r+b} = 0.01\,, r + b = 100, c = 1\,, \mu = 0.808080808$

x	$\mathrm{P_Y}(x)$	$B\ (x)_{n,\frac{r}{r+b}}$	$\widetilde{\wp_\lambda(x)}$	$\wp_\lambda(x)$ $\mu = n\frac{r}{r+b}$	$\lvert \mathrm{P_Y}(x) - \widetilde{\wp_\lambda(x)} \rvert$	$\left\lvert B\ (x)_{n,\frac{r}{r+b}} - \widetilde{\wp_\lambda(x)} \right\rvert$	$\left\lvert B\ (x)_{n,\frac{r}{r+b}} - \wp_\lambda(x) \right\rvert$	$\lvert \mathrm{P_Y}(x) - \wp_\lambda(x) \rvert$
0	No Result	0.44752321	0.44752321	0.44571265	No Result	0.00000000	0.00181056	No Result
1	No Result	0.36163492	0.36163492	0.36017184	No Result	0.00000000	0.00146308	No Result
2	No Result	0.14428868	0.14431123	0.14552398	No Result	0.00002255	0.00123530	No Result
3	No Result	0.03789400	0.03793504	0.03919838	No Result	0.0000041 05	0.00130438	No Result
4	No Result	0.00736828	0.00739631	0.00791886	No Result	0.00002803	0.00055059	No Result
5	No Result	0.00113129	0.00114224	0.00127982	No Result	0.00001094	0.00014853	No Result

Example 4.2 : Suppose $n = 10\,, r + b = 1000, \frac{r}{r+b} = 0.01 \quad \mu = 0.101010101$**, then** numerical results are as follows

x	$\mathrm{P_Y}(x)$	$B\ (x)_{n,\frac{r}{r+b}}$	$\widetilde{\wp_\lambda(x)}$	$\wp_\lambda(x)$ $\mu = n\frac{r}{r+b}$	$\lvert \mathrm{P_Y}(x) - \widetilde{\wp_\lambda(x)} \rvert$	$\left\lvert B\ (x)_{n,\frac{r}{r+b}} - \widetilde{\wp_\lambda(x)} \right\rvert$	$\left\lvert B\ (x)_{n,\frac{r}{r+b}} - \wp_\lambda(x) \right\rvert$	$\lvert \mathrm{P_Y}(x) - \wp_\lambda(x) \rvert$
0	0.904790682	0.904382075	0.903923902	0.904837418	0.000408607	0.000000000	0.000455343	0.000004674
1	0.090569634	0.091351724	0.091351724	0.090483742	0.000782090	0.000000000	0.000867982	0.000085892
2	0.004492181	0.004152351	0.004194294	0.004524419	0.000297887	0.000041943	0.000371836	0.000032006
3	0.000144182	0.000111847	0.000119496	0.000150806	0.000024686	0.000007648	0.000038958	0.000006624
4	0.000000329	0.000001977	0.000002452	0.000003770	0.000000841	0.000000475	0.000001793	0.000000477

5	0.000000056	0.000000024	0.000000040	0.000000075	0.000000016	0.000000016	0.000000051	0.000000019

Example 4.3 : Suppose $n = 300$, $r + b = 100, \frac{r}{r+b} = 0.01$ $\mu = 3.03030303$, then numerical results are as follows

x	$P_Y(x)$	$B(x)_{n,\frac{r}{r+b}}$	$\widetilde{\wp_\lambda(x)}$	$\wp_\lambda(x)$ $\mu = n\frac{r}{r+b}$	$\lvert P_Y(x) - \widetilde{\wp_\lambda(x)}\rvert$	$\lvert B(x)_{n,\frac{r}{r+b}} - \widetilde{\wp_\lambda(x)}\rvert$	$\lvert B(x)_{n,\frac{r}{r+b}} - \wp_\lambda(x)\rvert$	$\lvert P_Y(x) - \wp_\lambda(x)\rvert$
0	No Result	0.04904089	0.04904089	0.04830100	No Result	0.00000000	0.00073989	No Result
1	No Result	0.14860877	0.14860877	0.14636666	No Result	0.00000000	0.00224211	No Result
2	No Result	0.22441425	0.22441675	0.22176767	No Result	0.00000249	0.00265658	No Result
3	No Result	0.22516986	0.22518732	0,22400775	No Result	0.00001746	0.00116211	No Result
4	No Result	0.16887739	0.16892394	0.16970284	No Result	0.00004655	0.00082545	No Result
5	No Result	0.10098527	0.10105714	0.10285021	No Result	0.00007187	0.00186494	No Result

Example 4.4: Suppose $n = 10, r = 5, r + b = 100$, $\frac{r}{r+b} = 0.05, \mu = 0.526315789$

x	$P_Y(x)$	$B(x)_{n,\frac{r}{r+b}}$	$\widetilde{\wp_\lambda(x)}$	$\wp_\lambda(x)$ $\mu = n\frac{r}{r+b}$	$\lvert P_Y(x) - \widetilde{\wp_\lambda(x)}\rvert$	$\lvert B(x)_{n,\frac{r}{r+b}} - \widetilde{\wp_\lambda(x)}\rvert$	$\lvert B(x)_{n,\frac{r}{r+b}} - \wp_\lambda(x)\rvert$	$\lvert P_Y(x) - \wp_\lambda(x)\rvert$
0	0.612207502	0.598736939	0.598736939	0.606530660	0.013470563	0.0000000	0.007793721	0.005676842
1	0.294330530	0.315124705	0.315124705	0.303265330	0.020794175	0.0000000	0.011859375	0.008934800
2	0.077154605	0.074634799	0.075388683	0.075816332	0.001765922	0.000753884	0.001181533	0.001338273
3	0.016265635	0.010475089	0.011191303	0.012636055	0.005074332	0.000716214	0.002160966	0.003629580
4	0.001957200	0.000964805	0.001196439	0.001579507	0.000760761	0.000231634	0.000614702	0.000377693
5	0.000211378	0.000060935	0.000100753	0.000157951	0.000110625	0.000039818	0.000053427	0.000053427

5.0 DISCUSSION

From example 4.1-4.3 , it was found that the new improved Poisson is sufficient enough to approximate Binomial more the normal Poisson distribution provided $\frac{r}{r+b}$ is small , but not sufficient to approximate Polya distribution.

6. CONCLUSION

In this work , the new improved Poisson distribution with $\mu = \frac{rr}{b}$ is obtained from Polya distribution with parameters r, b, n and c. The result obtained gives a good approximation to Binomial distribution provided $\frac{r}{r+b}$ is small, but not sufficient enough to give a good approximation to Polya distribution. By comparison new improved Poisson remain the best approximation to Binomial and while normal Poisson remains the best approximation to Polya distribution more the new improved Poisson distribution .

References

[1] Bright O.Osu ,Samson O.Egege and Silas Ihedioha "Option Evaluation : Black Scholes Model versus Improved Poisson Model in Option Pricing" Transaction on Nigeria Association of Mathematics Physics (NAMP) vol7, pp55-62, 2016

[2] Dongping Hu, Yongquan Cui, and Aihua Yin "An improved Negative Binomial Approximation for Negative Hypergeometric Distribution", Applied Mechanice and Material, vol 427-429,pp 106-110,2013

[3] Samson O.Egege ,Bright O. Osu and Emmanuel J. Ekpeyong "Application of generalized Binomial model in option pricing ,American Journal of Applied Mathematics and Statistics ,vol5, N0 2 pp 62-71, 2017

[4] Samson O. Egege , Bright .O. Osu and Chigozie Chibuisi "An Improved Poisson Distribution and Its Application in Option Pricing" , Open Science Journal of Mathematics and Application vol 6, No3 pp 15-22 ,2018

[5] Samson O. Egege ,Bright O.Osu and Chigozie Chibuisi "A non Uniform bound approximation of Polya via Poisson , using Stein's Chen method and w-function and its application in option pricing", International journal of Mathematics and Statistics Invention vol 6.Issue 3 pp 09-20, 2018

[6] Samson O. Egege , Bright O. Osu , Kingsley Uchendu and Chiemela B. Akachi "An improved Poisson approximation for the generalized Binomial distribution with financial Application" Elixir International journal of Applied Mathematics , 121, pp 51509-51519 , 2018

[7] Samson O.Egege**,** Bright O. Osu and Chogozie Chibuisi. "A non- uniform bound approximation of Pólya via Poisson,Using ,Stein –Chen method and W- function and its application in option pricing". International journal of Mathematics and Statistics Invention (IJMSI) Vol 6 Issue 3 ,PP 09-20, 2018

[8] Teerapabolarn K. and Jaioun K. "A improved Poisson Approximation for Binomial Distribution "Applied Mathematics Sciences , vol 8, no 174,pp 8651-8654,2014

[9] Teerapabolarn K "Approximation of Binomial Distribution by an Improved Poisson Distribution" International Journal of Pure and Applied Mathematics vol 97,n0 4 pp 491-495 , 2014

[10] Teerapabolarn K "An improved Poisson Aproximation to Binomial Distribution" International journal of Mathematics Trend and Technology , vol 68,issue 9, pp16-20 ,2022

[11] Teerapabolarn K "Binomial Approximation to the Polya Distribution" international journal of Pure and Applied Mathematics ,vol 78 N0.5 pp 635-640 , 2012